Perturbation Method of Structural Design Relevant to Holographic Vibration Analysis

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Holographic vibration analysis provides a rapid way of obtaining accurate data on the shapes of normal vibration modes. At present, these data can be used to change the design of a structure only if a theoretical solution for the structure's vibration modes is known, either in closed form or by finite element analysis. A method is presented here to bypass formal solutions for the normal modes of a structure in predicting the results of small changes to a prototype, when normal-mode data, such as that obtained by hologram interferometry, can be used as a starting point. From the known mode shapes and frequencies of a structure, it is possible to calculate new mode shapes and frequencies by first-order perturbation, given small changes in mass distribution and stiffness moduli. An optimum design could be obtained by repeating the process a number of times with successively smaller changes being made to the structure.

Introduction

IN a previous discussion of normal mode theory with respect to holographic vibration analysis,[1] the conclusion was drawn that the amount of data needed to analyze a structure could be minimized if it were possible to make holographic recordings of the structure while it vibrates in each of its normal vibrational modes, one at a time. Any vibrational state of a structure should be capable of being represented by a series of these mode functions. Furthermore, such a representation has been shown to be applicable to static deflections.[2] The mode functions can be obtained from the location of the dark fringes in the reconstructions of holograms recorded while the structure vibrates in its normal modes.

In many situations, a vibration analysis of a structure is only the first step toward the modification of the structure to achieve some goal. Consider, for example, "the fiddlemaker's problem." A fiddlemaker's best violin may sell for 100 times less than one made by Stradivarius. If he were given the vibration-mode data of his best violin, and that of a famous Stradivarius violin, it would only frustrate him unless he knew what to do to his violin to change its modes to match those of the Stradivarius. The violin is a very illustrative example because it is a complex structure made of anisotropic material, whose qualities vary widely with the source, and is sensitive to environmental humidity. These and other aspects of the violin make it a very difficult structure to analyze formally, even by finite element methods that use digital computers. Although structures of an engineering interest are usually made of materials more consistent than wood, there may exist complexities that make finite-element analyses expensive as well as difficult. In addition, composite materials are coming into more frequent use, and their properties offer a challenge to the existing methods of theoretical analysis. Finally, there is the problem of what to do with a prototype that has been built, after an analysis has been performed, and which does not behave exactly as predicted.

Hologram interferometry has progressed in recent years to the point where recording data on vibration modes can be done quickly at a very modest cost. In fact, the amount of data usually obtained is far in excess of current capacities to make use of it. The foregoing issues make it of interest to consider design methods that use an abundance of experimental data to provide an alternative to theoretical methods that proceed from first principles in calculating vibration modes. An obvious method is that of perturbation. It has long been a common technique to use perturbation methods to describe the solution of problems that depart only slightly from problems whose formal solutions are known. Such methods, however, become inaccurate if the problem departs too greatly from the one that can be solved. This difficulty can be overcome if a set of solutions can be experimentally provided to describe a problem from which an arbitrarily small departure is to be made. Hologram interferometry can provide such a starting point at any step in the evolution of a structural design, and perturbation procedures could be used iteratively to achieve a design goal. To this end, this paper will present a perturbation analysis for the vibration modes of a structure, that will express a new set of vibration-mode functions and mode frequencies in terms of a previously known set and a prescribed change in the structure. The analysis will not be based upon specific differential equations for specific types of structures (e.g., plates, shells, or membranes), but rather upon orthogonality and energy relationships. As a consequence, it might be of considerable value if it were converted to analysis schemes that involve finite elements; however, for this paper only continuous structures and continuous mode functions are considered.

Primary Equations

To begin, let us assume that the structure has no damping and that its motions may be characterized by a set of normal mode functions Φ_n, each of which is associated with a natural frequency ω_n, expressed in radians per second. If the structure vibrates sinusoidally in only one of these normal modes, its motion will be uniphase and the maximum kinetic energy (when moving through rest position) will equal the maximum potential energy (when at an extreme point of deflection).[3] This may be expressed as†

$$\omega_n^2 M_n = K_n \tag{1}$$

where

$$M_n = \int m(x)\Phi_n^2(x)\,dx \tag{2}$$

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† Note that the kinetic and potential energies have been multiplied by 2 in Eq. (1) to remove the factors of $\frac{1}{2}$ that occur in the energy calculations.

$m(x)$ is the mass per unit volume of the structure, x denotes *three-dimensional* space, and the integration is carried out over the volume of the structure. The parameter M_n is often called the generalized mass of the structure with respect to the mode Φ_n; however, in the present context it will be referred to as the nth modal mass of the structure. The right-hand member of Eq. (1), K_n, is the potential energy stored in the structure when it is bent to the shape of Φ_n. Let us represent this by

$$K_n = \int \Sigma_q K_q(x) D_q\{\Phi_n(x)\}\, dx \tag{3}$$

where Σ_q indicates summation with respect to the index q, $K_q(x)$ are a set of stiffness moduli, and $D_q\{\ \}$ are a set of nonlinear, differential operators. The parameter K_n is often called the generalized stiffness of the structure with respect to the mode Φ_n; however, in the present context it shall be called the nth modal stiffness of the structure.

It is a general property of normal mode functions that they are mutually orthogonal with respect to mass density as a weighting factor. Thus

$$\int m(x)\Phi_n(x)\Phi_k(x)\, dx = 0 \tag{4}$$

where $\Phi_n(x)$ and $\Phi_k(x)$ are two normal mode functions of the structure. From Rayleigh's principle,[4] we may state that if Φ_n is replaced by $\Phi_n + C\Phi_k$ in Eqs. (2) and (3) and substituted into the right side of Eq. (1), the resulting quotient will vary as C^2. (C is a small constant, much less than 1.0.) This replacement in Eq. (2) leads directly to

$$\int m[\Phi_n + C\Phi_k]^2\, dx = M_n + C^2 M_k \tag{5}$$

and therefore M_n also varies as C^2. This, in turn, implies that K_n also must vary as C^2 for the same substitution, and this leads to

$$\lim_{C\to 0}(C^{-1})\int \Sigma_q K_q[D_q\{\Phi_n + C\Phi_k\} - D_q\{\Phi_n\}]\, dx = 0 \tag{6}$$

where the linear variation of K_q with respect to C has been set to zero. As C approaches zero, Eq. (6) approaches

$$\int \Sigma_q K_q D'_q\{\Phi_n\}\Phi_k\, dx = 0 \tag{7}$$

where $D'_q\{\Phi_n\}$ is the derivative of D_q evaluated at Φ_n. (This is an example of a Gateaux derivative).[5] This derivative operating on Φ_n is a linear operator which operates upon Φ_k. For a large class of vibration problems, it is possible to restrict the differential operators D_q to 1) the squares of linear differential operators (i.e., $[L]^2$, where $L\{\ \}$ is a linear differential operator), or 2) the product of two linear differential operators (i.e., $L_a\{\ \}\, L_b\{\ \}$. For these two cases,

$$D'_q\{\Phi_n\}\Phi_k = 2L\{\Phi_n\}L\{\Phi_k\} \tag{8}$$

when $D_q\{\Phi\} = [L\{\Phi\}]^2$ and

$$D'_q\{\Phi_n\}\Phi_k = L_a\{\Phi_n\}L_b\{\Phi_k\} + L_a\{\Phi_k\}L_b\{\Phi_n\} \tag{9}$$

when $D_q\{\Phi\} = L_a\{\Phi\}L_b\{\Phi\}$. Thus, $D'_q\{\Phi_n\}\Phi_k$ may be replaced by $D_{l_q}\{\Phi_n, \Phi_k\}$, where $D_{l_q}\{\ \}$ is a symmetrical joint-differential operator that is a linear operator upon Φ_n as well as upon Φ_k. This restriction together with that imposed by the definition of Eq. (3) define the class of vibration problems that are considered by this perturbation theory. Equation (7) can be rewritten as

$$\int \Sigma_q K_q(x) D_{l_q}\{\Phi_n(x), \Phi_k(x)\}\, dx = 0 \tag{10}$$

Equations (1), (4), and (10) are the three that will be subject to perturbation. Before proceeding to this, let us substitute $\Phi_n + C\Phi_n$ for Φ_n into Eq. (1) via Eqs. (2) and (3). The result is

$$\omega_n^2 M_n + 2C\omega_n^2 M_n + C^2\omega_n^2 M_n = K_n + C\int \Sigma_q K_q D_{l_q}\{\Phi_n, \Phi_n\}\, dx + \ldots \tag{11}$$

from which it may be concluded that

$$2K_n = \int \Sigma_q K_q D_{l_q}\{\Phi_n, \Phi_n\}\, dx \tag{12}$$

First-Order Perturbations

Let us now introduce small changes in the mass and in the stiffness moduli. The result is a modified structure that will be characterized by a modified set of vibration modes. The frequency of each new mode will be slightly different from the frequency of each corresponding old mode, and the shape of each new mode will be slightly different from that of each corresponding old mode. Thus the following substitutions can be made:

$$m'(x) = m(x) + \Delta m(x) \tag{13}$$

$$K'(x) = K(x) + \Delta K(x) \tag{14}$$

$$\omega'_n = \omega_n + \Delta\omega_n \tag{15}$$

$$\Phi'_n(x) = \Phi_n(x) + \Delta\Phi_n(x) = \Sigma_i C_{ni}\Phi_i(x),\ C_{nn} = 1 \tag{16}$$

where the primes denote the new structure. Equation (16) can be expressed conveniently in matrix form as

$$\begin{bmatrix}\Phi'_1\\ \vdots\\ \Phi'_n\end{bmatrix} = \begin{bmatrix}1 & \cdots & C_{1n}\\ \vdots & & \vdots\\ C_{n1} & \cdots & 1\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_n\end{bmatrix} \tag{17}$$

Let us substitute Eqs. (13–16) into Eq. (1) via Eqs. (2) and (3). The result, to first-order approximation, is

$$\omega_n^2 M_n + \Delta M_{nn}\omega_n^2 + 2\omega_n\Delta\omega_n M_n = K_n + \Delta K_{nn} + \int \Sigma_q K_q D_{l_q}\{\Phi_n, \sum_{i\neq n} C_{ni}\Phi_i\}\, dx \tag{18}$$

where

$$\Delta M_{nn} = \int \Delta m(x)\Phi_n^2(x)\, dx \tag{19}$$

$$\Delta K_{nn} = \int \Sigma_q \Delta K_q(x) D_q\{\Phi_n(x)\}\, dx \tag{20}$$

From Eq. (10), it can be shown that the integral in Eq. (18) is identically zero. Thus, Eq. (18) can be solved for the unknown $\Delta\omega_n$ to give

$$(\Delta\omega_n/\omega_n) \approx \tfrac{1}{2}(\Delta K_{nn}/K_n - \Delta M_{nn}/M_n) \tag{21}$$

Equation (21) expresses the change in frequency of each mode in terms of the change in mass, change in stiffness, and the original frequency and shape of the mode.

Now let us determine the coefficients of the matrix in Eq. (17), that express the change in shape of each mode. The new mode shapes must be mutually orthogonal with the new mass distribution as a weighting factor; thus Eqs. (13) and (16) may be substituted into Eq. (4).

$$\int (m + \Delta m)(\Sigma_i C_{ni}\Phi_i)(\Sigma_j C_{kj}\Phi_j)\, dx = 0 \tag{22}$$

If this equation is expanded, Eq. (4) may be used to set to zero all integrals involving the products, $m(x)\Phi_i(x)\Phi_j(x)$. The result to first-order approximation is

$$\Delta M_{nk} + C_{kn} M_n + C_{nk} M_k \approx 0 \tag{23}$$

where

$$\Delta M_{nk} = \int \Delta m(x)\Phi_n(x)\Phi_k(x)\, dx \tag{24}$$

The new mode shapes must also satisfy Eq. (10) if the new stiffness moduli are substituted for the old ones. Thus,

$$\int \Sigma_q(K_q + \Delta K_q) D_{l_q}\{\Sigma_i C_{ni}\Phi_i, \Sigma_j C_{kj}\Phi_j\}\, dx = 0 \tag{25}$$

The differential operations in Eq. (25) may be expanded by making use of Eqs. (8) and (9) to give

$$D_{lq}\{\Sigma_i C_{ni}\Phi_i, \Sigma_j C_{kj}\Phi_j\} = \Sigma_i\Sigma_j C_{ni} C_{kj} D_{lq}\{\Phi_i, \Phi_j\} \tag{26}$$

Substitution into Eq. (25) and expansion gives, to first-order approximation,

$$\Delta K_{nk} + C_{kn} K_n + C_{nk} K_k \approx 0 \tag{27}$$

where

$$\Delta K_{nk} = \tfrac{1}{2}\int \Sigma_q \Delta K_q D_{lq}\{\Phi_n, \Phi_k\}\, dx \tag{28}$$

and where Eq. (10) was used to set to zero all integrals involving the integrands $\Sigma_q K_q(x) D_{lq}\{\Phi_i(x), \Phi_j(x)\}$. Equations (23) and (27) are linear equations in the two unknowns, C_{kn} and C_{nk}. Their simultaneous solution for C_{nk} is

$$C_{nk} \approx (\Delta M_{nk} K_n - \Delta K_{nk} M_n)/(M_n K_k - M_k K_n) \tag{29}$$

Using Eq. (1), Eq. (29) becomes

$$C_{nk} \approx (\omega_n^2 \Delta M_{nk}/M_k - \omega_k^2 \Delta K_{nk}/K_k)/(\omega_k^2 - \omega_n^2) \tag{30}$$

Discussion

The denominator of Eq. (30) allows a helpful statement as to the number of modes that must be considered in a perturbation calculation of this sort. Both Eqs. (16) and (17) have described

the new mode shapes as admixtures of the original mode shapes with the constants C_{nk} being the admixture coefficients. Phrasing it differently, the change in mass and stiffness of a structure causes each mode shape to subsume small amounts of all the other mode shapes. Equation (30) indicates that the admixture coefficients are inversely proportional to the difference of the squares of the resonant frequencies of the two admixing modes. Thus, each mode shape will subsume greater amounts of the shapes of those modes that lie closer in frequency to the mode considered. Therefore, in calculating the new shape of each mode, it may be sufficient to consider the finite number of modes within a certain frequency bandwidth surrounding the mode in question. With respect to calculating the new frequency of a mode, only the shape of the original mode need be considered, as can be seen from Eq. (21).

Equations (21) and (30) both have two terms in their right-hand members, one that is proportional to a change in mass and one that is proportional to a change in stiffness, and these are opposite in sign. Thus, a perturbation may be considered mass-like or spring-like depending upon which of the two terms has the larger magnitude. Equation (21) shows that an increase in stiffness will raise the resonant frequency of a mode whereas an increase in mass will lower it. The two terms in the right-hand member of Eq. (30) require a more detailed discussion, which follows.

Let us assume that the perturbation involves only a change in mass over a limited area. If a positive sign is given to each mode function in the region where the mass is changed (assuming no node lines in this region), then C_{nk} will be positive for an increase in mass and C_{kn} will be negative, if ω_n is less than ω_k. Physically this means that the lower frequency mode will increase its amplitude where the mass is added and the higher frequency mode will decrease its amplitude in the same region. The low frequency mode makes use of the added mass and the high frequency mode avoids use of the added mass. If the perturbation were purely a change in stiffness, the opposite would be true; the higher frequency mode would make use of the added stiffness and the lower frequency mode would avoid use of the added stiffness. Thus, for these cases, the sign of C_{nk} is opposite to that of C_{kn}.

In general, the magnitudes of the two complementary admixture coefficients are not equal and this may be examined by taking their ratio. From Eq. (29),

$$C_{nk}/C_{kn} = -(\Delta M_{nk} K_n - \Delta K_{nk} M_n)/(\Delta M_{nk} K_k - \Delta K_{nk} M_k) \qquad (31)$$

recognizing that $\Delta M_{nk} = \Delta M_{kn}$ and $\Delta K_{nk} = \Delta K_{kn}$. For purely mass perturbations, the magnitude of this ratio is less than one, whereas for purely stiffness perturbations it is equal to or greater than one. This is true because modal-mass is generally constant or decreases with mode frequency whereas modal-stiffness increases with frequency.

If the perturbation involves a change in both mass and stiffness, it is possible for the complementary coefficients to have the same sign, depending upon the relative strength of the two effects. It is also possible for one coefficient to be zero when the other is not. This seems somewhat paradoxical in that it is possible for one mode to take something of another without giving something of itself in return. However, in practice this may be a desirable feature when the shape of one mode is to be changed without affecting that of another. It must be kept in mind, however, that this discussion has ignored the fact that each mode will admix, in general, with all of the other modes, so that the results can be quite complicated.

Isotropic and Orthotropic Plates

Let us consider two specific examples of perturbation due to a change in material thickness. For isotropic and orthotropic plates, the stiffness moduli K_q are all proportional to the thickness of the plate cubed, and there are four differential operators $D_q\{\ \}$ if the middle surface of the plate is not under stress. For the isotropic case,[6] Eq. (20) becomes

$$\Delta K_{nn} = E/4(1-\nu^2) \int h^2(x,y) \Delta h(x,y) \times [(\Theta_n{}^{xx})^2 + (\Theta_n{}^{yy})^2 + 2\nu \Theta_n{}^{xx} \Theta_n{}^{yy} + 2(1-\nu)(\Theta_n{}^{xy})^2]\, dx\, dy \qquad (32)$$

where E is Young's modulus, ν is Poisson's ratio, $h(x,y)$ is the thickness of the plate, and the superscripts are used to denote partial derivatives with respect to the superscript variable. Correspondingly, Eq. (19) becomes

$$\Delta M_{nn} = m \int h(x,y) \Theta_n{}^2(x,y)\, dx\, dy \qquad (33)$$

where m is the volume density of the material. The joint operators $D_{lq}\{\ \}$ can be derived from Eqs. (8) and (9), and thus Eq. (28) becomes

$$\Delta K_{nk} = E/4(1-\nu^2) \int h^2 \Delta h \times [\Theta_n{}^{xx} \Theta_k{}^{xx} + \Theta_n{}^{yy} \Theta_k{}^{yy} + \nu(\Theta_n{}^{xx} \Theta_k{}^{yy} + \Theta_n{}^{yy} \Theta_k{}^{xx}) + 2(1-\nu) \Theta_n{}^{xy} \Theta_k{}^{xy}]\, dx\, dy \qquad (34)$$

and Eq. (24) becomes

$$\Delta M_{nk} = m \int h(x,y) \Theta_n(x,y) \Theta_k(x,y)\, dx\, dy \qquad (35)$$

The parameters, ΔK_{nn}, ΔM_{nn}, ΔK_{nk}, and ΔM_{nk}, now may be substituted into Eqs. (21) and (30) to yield the change in frequency of each mode and the coefficient of admixture.

For the orthotropic case, Eq. (20) becomes

$$\Delta K_{nn} = (\tfrac{1}{4}) \int h^2 \Delta h \times [E'_x(\Theta_n{}^{xx})^2 + E'_y(\Theta_n{}^{yy})^2 + 2E'' \Theta_n{}^{xx} \Theta_n{}^{yy} + 4G(\Theta_n{}^{xy})^2]\, dx\, dy \qquad (36)$$

where E'_x, E'_y, E'', and G are four experimentally determined material constants.[7] Correspondingly, Eq. (28) becomes

$$\Delta K_{nk} = (\tfrac{1}{4}) \int h^2 \Delta h \times [E'_x \Theta_n{}^{xx} \Theta_k{}^{xx} + E'_y \Theta_n{}^{yy} \Theta_k{}^{yy} + E''(\Theta_n{}^{xx} \Theta_k{}^{yy} + \Theta_n{}^{yy} \Theta_k{}^{xx}) + 4G \Theta_n{}^{xy} \Theta_k{}^{xy}]\, dx\, dy \qquad (37)$$

Equations (33) and (35) remain unchanged. Thus it can be seen that formalistically there is little difficulty in carrying this analysis from isotropic plates to orthotropic plates.

These two examples illustrate an important point with respect to perturbation calculations on the thickness variations of structures. In order to calculate the parameters ΔK_{nn} and ΔK_{nk}, it is necessary to calculate the second partial derivatives of the mode functions from experimentally measured values of the amplitudes of the mode functions. Investigations into this problem have been carried out for the purpose of strain analysis,[8] and it may be expected that the techniques developed may be applied to perturbation calculations as well.

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